

VALUE FUNCTIONS ON CENTRAL SIMPLE ALGEBRAS

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ABSTRACT. In this paper we study noncommutative valuation rings as defined by Dubrovin. While there is in general no valuation associated to a Dubrovin valuation ring, we show that there is a value function associated to any Dubrovin valuation ring integral over its center. By using value functions we obtain information on Dubrovin valuation rings in a tensor product, both generalizing and giving a much simpler proof of a result about valued division algebras. By being able to work directly with central simple algebras we gain new information about division algebras over Henselian fields.

1. INTRODUCTION

In recent years the study of valuations on division algebras has proven useful in solving problems in the theory of algebras. For instance, valued division algebras were used in the construction of noncrossed products $[A, \text{JW}_1, T]$ and in the construction of division algebras with nontrivial SK_1 $[P, DK]$. The Malcev-Neumann algebras studied by Tignol and Amitsur in $[TA]$ are further examples of valued division algebras. Work on valued division algebras has intensified somewhat in the last few years, and a reasonable amount is known about such algebras (see for instance $[\text{JW}_2, \text{PY}, \text{TW}]$).

There are some problems in dealing with valued division algebras. One problem in particular is how to generalize the notion of valuation to matrix algebras. Matrix algebras arise naturally when dealing with division algebras, for instance as the tensor product of two division algebras. Having a "valuation" on a matrix algebra that yields information about the underlying division algebra of the algebra would be a helpful tool, since getting at the underlying division algebra can be quite difficult in general. In Van Geel's book $[V]$ there are a number of valuation ring and valuation-like conditions. A further candidate for valuation rings in matrix algebras is the rings Dubrovin defined in $[D_1]$. The nice structure and good extension and uniqueness properties of Dubrovin valuation rings (see $[\text{W}_1]$) indicate that these rings are perhaps the correct definition of a valuation ring of a simple Artinian ring.

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In this paper we introduce the notion of a value function on a simple Artinian ring. While there is no value function associated to an arbitrary Dubrovin valuation ring, we will prove that there is a value function associated to a Dubrovin valuation ring which is integral over its center. Value functions behave similarly to valuations on division algebras. We will prove a number of results about integral Dubrovin valuation rings (that is, Dubrovin valuation rings integral over their center) by using value functions in much the same way as one would use a valuation. For instance, by using value functions we are able to get information about Dubrovin valuation rings in a tensor product (Theorem 3.3). This leads to new information about division algebras over Henselian fields (see Corollary 3.6 and the comments preceding it). In [MW] this author and A. Wadsworth will use value functions to study valued division algebras. By being able to work directly with matrix algebras we will give new and simpler proofs of some results in [JW₂]. For the rest of this section we will give some definitions and elementary properties.

Let D be a division algebra with $F = Z(D)$, the center of D . We will only be considering finite dimensional algebras, i.e. $[D : F] < \infty$. A subring R of D is called an *invariant valuation ring* of D provided that for all $d \in D^*$,

- (a) $d \in R$ or $d^{-1} \in R$,
- (b) $dRd^{-1} = R$.

We will denote by T^* the group of units of a ring T . Property (b) shows that any one-sided ideal of R is actually two-sided, and (a) shows that the ideals of R are linearly ordered by inclusion. Thus the set $\Gamma = \{dR \mid d \in D^*\}$ can be made into a totally ordered group by setting $dR \cdot eR = deR$ and $dR \leq eR$ iff $dR \supseteq eR$. Invariant valuation rings are those rings which arise from a valuation on D . A *valuation* on D is a function $v: D^* \rightarrow \Delta$, where Δ is a totally ordered abelian group, such that for all $a, b \in D^*$,

- (1) $v(ab) = v(a) + v(b)$,
- (2) $v(a + b) \geq \min\{v(a), v(b)\}$ if $a + b \neq 0$.

This is the definition given in Schilling's book [S]. For convenience we extend v to D by setting $v(0) = \infty$, where $\infty > \delta$ and $\infty + \delta = \delta + \infty = \infty$ for all $\delta \in \Delta$. The connection between valuations and invariant valuation rings is as follows. If v is a valuation on D then $R_v = \{d \in D \mid v(d) \geq 0\}$ is easily seen to be an invariant valuation ring. Conversely, if R is an invariant valuation ring of D , then the map $v: D^* \rightarrow \Gamma$ given by $v(d) = dR$ for Γ as above is a valuation on D .

Suppose v is a valuation on D with corresponding invariant valuation ring R . Then $R^* = \ker(v)$, so $D^*/R^* \cong v(D^*)$. We call D^*/R^* the *value group* of R (or v) and denote it Γ_R . The Jacobson radical $J(R)$ of R is the set $\{d \in R \mid d = 0 \text{ or } d^{-1} \notin R\} = R - R^*$, so $J(R)$ is the unique maximal left (and right) ideal of R . Thus $\bar{R} = R/J(R)$ is a division ring, called the *residue division ring* of R .

As is well known (e.g. [E, 7.1, 8.6]) for commutative fields the notions of valuation, valuation ring and place are all equivalent. Dubrovin's rings correspond to the idea of a place on a simple Artinian ring. These rings are defined as follows. A subring B of a simple Artinian ring S is called a *Dubrovin valuation ring* of S provided that there is an ideal J of B such that

(a) B/J is simple Artinian,

(b) If $s \in S - B$ then there are $b_1, b_2 \in B$ with $b_1 s, s b_2 \in B - J$.

The place associated to B is the map $S \rightarrow B/J \cup \{\infty\}$ given by $s \mapsto s + J$ if $s \in B$ and $s \mapsto \infty$ otherwise. It turns out that $J = J(B)$ and that B is a two-sided order in S . Thus B is a prime Goldie ring. Furthermore, if $F = Z(S)$ then $Z(B) = B \cap F$ is a valuation ring of F . Dubrovin proved the following properties, which justify calling these rings valuation rings.

(1) The two-sided ideals of B are linearly ordered by inclusion, although the one-sided ideals are generally not linearly ordered.

(2) Finitely generated one-sided ideals are principal and projective as B -modules.

(3) If A is an overring of B in S , then A is a Dubrovin valuation ring of S and $J(A)$ is a prime ideal of B . If $[S : F] < \infty$ then there is a 1-1 correspondence between the prime ideals of B , prime ideals of $Z(B)$ and overrings of B in S . Furthermore if $P = J(A) \cap Z(B)$, then $A = B_P$, the central localization of B at P .

(4) If C is a subring of B containing $J(B)$ then C is a Dubrovin valuation ring of S iff $C/J(B)$ is a Dubrovin valuation ring of $B/J(B)$.

(5) The $n \times n$ matrix ring $M_n(B)$ is a Dubrovin valuation ring in $M_n(S)$. Conversely, if A is a Dubrovin valuation ring of $M_n(S)$ then there is a Dubrovin valuation ring A_0 of S and an $x \in M_n(S)^*$ with $A = x M_n(A_0) x^{-1}$.

Proofs of these statements can be found in [D₁, §1, Theorems 4 and 7; §2, Theorem 4] and [D₂, §1, Proposition 2; §2, Theorem 1]. Examples of Dubrovin valuation rings include invariant valuation rings, matrices over invariant valuation rings and Azumaya algebras over commutative valuation rings [D₂, §2, Proposition 1]. If V is a discrete valuation ring of a field F and S is central simple F -algebra then a subring B of S is a Dubrovin valuation ring of S with $B \cap F = V$ iff B is a maximal order over V [W₂, Ex. 1.15]. The literature on Dubrovin valuation rings consists of the papers [D₁, D₂, BG, W₁ and W₂].

For a Dubrovin valuation ring B of S , there is a *residue ring* $\bar{B} = B/J(B)$, a simple Artinian ring. Furthermore, as in [W₂, p. 304], we can define a value group. If $st(B) = \{s \in S^* \mid sBs^{-1} = B\}$ then $st(B)$ is the normalizer of B^* in S^* . We then define the *value group* $\Gamma_B = st(B)/B^*$. Dubrovin proves in [D₂, §2, Proposition 4] that the monoid of two-sided ideals of B is commutative, hence Γ_B is an abelian group. If S is a division ring and B an invariant valuation ring of S then $st(B) = S^*$, so this definition of Γ_B is the same as the one given above. If $F = Z(S)$ and $V = B \cap F$ then as noted above V is a

valuation ring of F . It is easy to see that $J(B) \cap V = J(V)$ and $B^* \cap V = V^*$. Thus there are natural inclusions $\Gamma_V \subseteq \Gamma_B$ and $\bar{V} \subseteq Z(\bar{B})$.

There are two integer invariants t_B and n_B which help to understand and classify B . For example, $n_B = t_B$ iff B is integral over V [W_2 , Theorem F]. They are defined as follows. First, since \bar{B} is simple Artinian, by Wedderburn's theorem, \bar{B} is a matrix ring over a division ring. Let t_B be the matrix size of \bar{B} . For n_B , if (F_h, V_h) is the Henselization of (F, V) , then n_B is the matrix size of $S \otimes_F F_h$. Wadsworth proves in [W_2 , Theorem D] that t_B divides n_B and gives a formula for $s_B = n_B/t_B$ which lends itself to calculation. Furthermore, there is a version of the "Ostrowski theorem" [W_2 , Theorem C]:

$$(1) \quad [S : F] = |\Gamma_B : \Gamma_V| [\bar{B} : \bar{V}] s_B^2 \delta(B)$$

where $\delta(B)$ is called the *defect* of B and is 1 if $\text{char}(\bar{V}) = 0$ and is a power of $\text{char}(\bar{V})$ when this characteristic is prime. B is said to be *defectless* when $\delta(B) = 1$. We sometimes write $\delta(B/V)$ for $\delta(B)$ to indicate what the base ring is. As in the commutative (and invariant) case we call $e_{B,V} = |\Gamma_B : \Gamma_V|$ the *ramification index* and $f_{B,V} = [\bar{B} : \bar{V}]$ the *residue degree* of B/V . It is proven in [M_1 , Theorem 2] that B is invariant iff $n_B = 1$, and so (1) reduces to the Ostrowski theorem proved in [M_1 , Theorem 3] and in [Dr, Theorem 2] for F Henselian.

For commutative valuation rings and invariant valuation rings a significant use of the value group is to classify the ideals. If R is an invariant valuation ring of D then the elements of Γ_R are in 1-1 correspondence with the principal ideals of the form dR for $d \in D^*$. Since all R -ideals (that is R - R submodules of D) are sums of ideals of the form dR , as $dR = Rd$, Γ_R determines all R -ideals. For a Dubrovin valuation ring B of S this reasoning no longer holds since it is not true that any one-sided ideal of B is two-sided. The elements of Γ_B are in 1-1 correspondence with the ideals of the form $sB = Bs$ (so $s \in st(B)$), and so Γ_B classifies those B -ideals of the form $\sum_{s \in \mathcal{S}} sB$, where \mathcal{S} is a subset of $st(B)$. In general, not all B -ideals are of this form; indeed, all B -ideals are of this form precisely when B is integral over its center [M_2 , Theorem 3.1 or W_2 , Theorem F]. For B integral this property of B -ideals is what allows us to define a value function on S (see Definition 2.1). Integral Dubrovin valuation rings seem to behave the most like invariant valuation rings, and the presence of a value function allows many results about invariant valuation rings to be carried over to integral Dubrovin valuation rings.

One way Dubrovin valuation rings can be used to study valued division algebras is to help answer tensor product questions. For instance, suppose D_1 and D_2 are F -central division algebras with $R_i \subset D_i$ invariant valuation rings and $R_1 \cap F = R_2 \cap F = V$. Since $D_1 \otimes_F D_2$ is a central simple F -algebra, $D_1 \otimes_F D_2 \cong M_n(D)$ for some F -central division algebra D . Suppose D contains an invariant valuation ring R with $R \cap F = V$. In general, getting at R can be quite difficult. However, $B = M_n(R)$ is a Dubrovin valuation ring of

$D_1 \otimes_F D_2$ with $\Gamma_B = \Gamma_R$ and $\bar{B} = M_n(\bar{R})$ [W₂, Corollary 3.5]. In a number of occasions in [MW] we construct a Dubrovin valuation ring B' in $D_1 \otimes_F D_2$ with $B' \cap F = V$ without too much difficulty. The following theorem, proved by Brungs and Gräter [BG, Theorem 5.4] for V of finite rank and by Wadsworth [W₂, Theorem A] in general shows $B' \cong M_n(R)$.

Theorem 1.1. *If S is a central simple F -algebra with B_1 and B_2 Dubrovin valuation rings of S and $B_1 \cap F = B_2 \cap F = V$, then $B_2 = sB_1s^{-1}$ for some $s \in S^*$.*

Thus we can determine Γ_R and \bar{R} by determining B' . In this paper and in [MW] such a B' is found by observing that on $D_1 \otimes_F D_2$ there is a naturally occurring value function which gives rise to B' just as a valuation gives rise to an invariant valuation ring.

2. VALUE FUNCTIONS

In this section we will define value functions and prove some basic properties. The results of the next section and [MW] will make heavy use of value functions.

Definition 2.1. *Let S be a simple Artinian ring and Γ a totally ordered abelian group. A function $w: S \rightarrow \Gamma \cup \{\infty\}$ is called a value function on S provided that for all $s, t \in S$,*

- (1) $w(s) = \infty$ iff $s = 0$ and $w(-1) = 0$,
- (2) $w(s + t) \geq \min\{w(s), w(t)\}$,
- (3) $w(st) \geq w(s) + w(t)$,
- (4) $\text{im}(w) = w(st(w))$, where $st(w) = \{s \in S^* \mid w(s^{-1}) = -w(s)\}$.

The above definition was motivated by the following example: let D be a division algebra and v a valuation on D with valuation ring R . If $S = M_n(D)$, we define a function $w: S \rightarrow \Gamma_R \cup \{\infty\}$ by

$$w((d_{ij})) = \min_{i,j} \{v(d_{ij})\}.$$

It is easy to see that w satisfies the conditions of Definition 2.1. This type of example can be found in [K]. By using value functions Kupferoth obtains a valuation on the universal division algebra $UD(\mathbf{Q}, n, t)$. It was recently pointed out to us that Benz studied value functions with value group a subgroup of \mathbf{R} in [Be].

The difference between value functions and valuations shows up in condition (3) of the above definition. This less restrictive multiplicative condition allows matrix algebras to have value functions, as seen by the above example. While it would appear that (3) is not a very restrictive condition, we shall see in Theorem 2.4 and Proposition 2.6 that this is not the case, and that equality in (3) holds sufficiently often for value functions to be an effective tool.

Lemma 2.2. *Let w be a value function on S . Then*

- (1) *If $s \in st(w)$ and $t \in S$ then $w(st) = w(ts) = w(s) + w(t)$.*

- (2) $st(w)$ is a subgroup of S^* and $w: st(w) \rightarrow \Gamma$ is a homomorphism.
 (3) If $w(s) \neq w(t)$ then $w(s+t) = \min\{w(s), w(t)\}$.
 (4) $B_w = \{s \in S \mid w(s) \geq 0\}$ is a ring and $J_w = \{s \in S \mid w(s) > 0\}$ is a two-sided ideal of B_w .

Proof. As $w(t) = w(s^{-1}st) \geq w(s^{-1}) + w(st) \geq -w(s) + w(s) + w(t) = w(t)$, we see equality holds everywhere, giving (1). If $s, t \in st(w)$ then $w(t) = -w(t^{-1})$, showing $t^{-1} \in st(w)$. Then by (1), $w((st)^{-1}) = w(t^{-1}s^{-1}) = w(t^{-1}) + w(s^{-1}) = -w(t) - w(s) = -(w(s) + w(t)) = -w(st)$. Thus $st(w)$ is a subgroup of S^* and by (1) again, $w: st(w) \rightarrow \Gamma$ is a homomorphism.

For (3) we first note that $-1 \in st(w)$ since $w((-1)^{-1}) = w(-1) = 0 = -w(-1)$. Thus for all $s \in S$, $w(-s) = w(-1) + w(s) = w(s)$ by (1). Now if $w(s) > w(t)$ then $w(t) = w((s+t) + (-s)) \geq \min\{w(s+t), w(s)\}$. Since $w(s) > w(t)$, this implies $w(t) \geq w(s+t) \geq \min\{w(s), w(t)\} = w(t)$, giving $w(s+t) = \min\{w(s), w(t)\}$.

Finally, to prove (4), properties (2) and (3) of Definition 2.1 show B_w is closed under addition and multiplication, and since $w(-s) = w(s)$ for all $s \in S$, B_w is closed under subtraction. Furthermore, $1 \in B_w$ as $w(-1) = 0$. Therefore B_w is a subring of S . Using properties (2) and (3) again, J_w is seen to be a two-sided ideal of B_w . \square

If R is an invariant valuation ring of a division algebra D then there is a valuation v on D with $R = \{d \in D \mid v(d) \geq 0\}$ and $J(R) = \{d \in D \mid v(d) > 0\}$. The next theorem shows the existence of a value function on a simple Artinian ring where B_w (as in Lemma 2.2) is a Dubrovin valuation ring.

Theorem 2.3. *Let S be a simple Artinian ring and B a Dubrovin valuation ring of S . Suppose for all $s \in S - \{0\}$ there is an $\alpha \in st(B)$ with $BsB = \alpha B$. Then there is a value function w on S with $B = B_w$ and $J(B) = J_w$. Furthermore, for any such w , $st(w) = st(B)$, $B^* = \{s \in st(B) \mid w(s) = 0\}$ and w induces an isomorphism $\Gamma_B \rightarrow im(w)$.*

Proof. We define a value function $w: S \rightarrow \Gamma_B \cup \{\infty\}$ as follows. First, let $w(0) = \infty$. For $s \in S - \{0\}$, by hypothesis there is an $\alpha \in st(B)$ with $BsB = \alpha B$. Let $w(s) = \alpha B^* \in \Gamma_B$, i.e. $w(s)$ is the image of BsB in Γ_B . It is clear that w is well defined since $\alpha B = \beta B$ iff $\alpha B^* = \beta B^*$ in Γ_B . The definition of the ordering on Γ_B gives $BsB \subseteq BtB$ iff $w(s) \geq w(t)$. We now verify the conditions of Definition 2.1. Condition (1) is clear. For (2), suppose $w(s) \geq w(t)$. Then $BsB \subseteq BtB$, and so $s+t \in BtB$, hence $B(s+t)B \subseteq BtB$. Therefore $w(s+t) \geq w(t) = \min\{w(s), w(t)\}$. To see (3), since additive notation is being used for values of w , the element $w(s) + w(t)$ refers to the image of $(BsB)(BtB)$ in Γ_B . As $st \in (BsB)(BtB)$, $BstB \subseteq (BsB)(BtB)$, hence $w(st) \geq w(s) + w(t)$. For (4), it is clear that $im(w) = w(st(B))$, and if $\alpha \in st(B)$, $w(\alpha^{-1}) = -w(\alpha)$ since $w(\alpha^{-1}) + w(\alpha)$ corresponds to the ideal $\alpha^{-1}B\alpha B = \alpha^{-1}\alpha B = B$. Thus $st(B) \subseteq st(w)$, so $im(w) = w(st(w))$. Now to

see $B = B_w$ and $J(B) = J_w$, $BsB \subseteq B$ (resp. $BsB = B$) iff $w(s) \geq w(1) = 0$ (resp. $w(s) = 0$). Since $BsB = B$ iff $s \in B - J(B)$, it follows that $B = \{s \in S \mid w(s) \geq 0\}$ and $J(B) = \{s \in S \mid w(s) > 0\}$.

Now, let w be any value function on S with $B = B_w$ and $J(B) = J_w$. For $b \in B^*$, $b^{-1} \in B$, so $w(b^{-1}) \geq 0$. But $0 = w(1) = w(bb^{-1}) \geq w(b) + w(b^{-1})$, so $0 \geq -w(b^{-1}) \geq w(b) \geq 0$, thus $w(b) = 0$. If $s \in st(w)$ with $w(s) = 0$ then $w(s^{-1}) = 0$, so $s, s^{-1} \in B$, hence $s \in B^*$. Therefore $B^* = \{s \in st(w) \mid w(s) = 0\}$. If $s \in st(w)$ then $w(sbs^{-1}) = w(b)$, so $w(sbs^{-1}) \geq 0$ iff $w(b) \geq 0$. Therefore $sBs^{-1} = B$, so $s \in st(B)$. Conversely, if $s \in st(B)$ then there is an $s_0 \in st(w)$ with $w(s) = w(s_0)$. Then $w(ss_0^{-1}) = 0$ by Lemma 2.2, so $ss_0^{-1} \in (B - J(B)) \cap st(B) = B^* \subseteq st(w)$ by $[W_2, \text{Lemma 3.2}]$. So $s \in st(w)$, hence $st(w) = st(B)$. Finally, by Lemma 2.2, $w: st(B) \rightarrow im(w)$ is a homomorphism. It is easy to see this map has kernel B^* and is surjective, so $\Gamma_B = st(B)/B^* \cong im(w)$. \square

If S is a central simple algebra and B is a Dubrovin valuation ring of S , then B has the extra property of Theorem 2.3 iff B is integral over its center, by $[M_2, \text{Theorem 3.1 or } W_2, \text{Theorem F}]$. The applications below and in $[MW]$ deal exclusively with this finite dimensional situation.

An alternative approach to the above proof (and my original approach) in the case S is a central simple algebra is the following. By $[W_2, \text{Theorem F}]$ there is a Dubrovin valuation ring B_h in $S \otimes_F F_h$ with $B_h \cap F_h = V_h$ and $B_h \cap S = B$, where (F_h, V_h) is the Henselization of (F, V) . Since F_h is Henselian, $B_h \cong M_n(R)$ where R is the invariant valuation ring in the underlying division algebra D of $S \otimes_F F_h$ (by Theorem 1.1 above). Defining w on $S \otimes_F F_h$ (possibly modifying it to account for the isomorphism $B_h \cong M_n(R)$) as in the example after Definition 2.1 and restricting to S gives the desired value function on S .

Frequently (as will be seen below and in $[MW]$) one wants to find a Dubrovin valuation ring in a central simple algebra and has a naturally occurring value function on the algebra. It would be useful to know which Dubrovin valuation rings come from (or give rise to) value functions. The following is a converse to Theorem 2.3 and was proved by A. Wadsworth.

Theorem 2.4. *Let w be a value function on S . Suppose B_w/J_w is simple Artinian. Then B_w is a Dubrovin valuation ring of S with $J(B_w) = J_w$ and for all $s \in S - \{0\}$ there is an $\alpha \in st(B_w)$ with $B_wsB_w = \alpha B_w$.*

Proof. To show B_w is a Dubrovin valuation ring of S with $J(B_w) = J_w$, it suffices to show that if $s \in S - B_w$ then there are $b_1, b_2 \in B_w$ with $sb_1, b_2s \in B_w - J_w$. Now since $s \notin B_w$, $w(s) < 0$. There is a $t \in st(w)$ with $w(s) = w(t)$. Hence $w(t^{-1}) = -w(t) > 0$, so $t^{-1} \in B_w$. Then $w(st^{-1}) = w(t^{-1}s) = w(s) - w(t) = 0$, so $st^{-1}, t^{-1}s \in B_w - J_w$. Setting $b_1 = b_2 = t^{-1}$ shows that B_w is a Dubrovin valuation ring. Let $s \in S - \{0\}$. Then as

above there is a $t \in st(w)$ with $w(st^{-1}) = 0$ hence $st^{-1} \in B_w - J_w$. Thus $B_w st^{-1} B_w = B_w$ as $J_w = J(B_w)$ is the unique maximal ideal of B_w . The same argument as in Theorem 2.3 shows $st(w) \subseteq st(B)$, so $B_w s B_w = t B_w$. \square

We combine Theorems 2.3 and 2.4 in the finite dimensional case for a useful corollary.

Corollary 2.5. *Let S be a central simple algebra and B a Dubrovin valuation ring of S . Then there is a value function w on S with $B = B_w$ and $J(B) = J_w$ iff B is integral over its center.*

The final result of this section gives a number of properties of value functions on central simple algebras. In particular it relates a value function to the reduced norm Nrd on a central simple algebra S . For information on reduced norms, see [R, Chapter 2.9] or [J, §9]. Before proving the lemma we mention one property of Nrd . Suppose S is a central simple F -algebra, V an integrally closed subring of F and B a subring of S integral over V . If $b \in B$ then $Nrd(b) \in V$. Furthermore, for $b \in B$, $b \in B^*$ iff $Nrd(b) \in V^*$. To see this, let $f(x)$ be the reduced characteristic polynomial of b . As b is integral over V and V is integrally closed, the minimal polynomial $p(x)$ of b lies in $V[x]$, and since f divides a power of p , f lies in $V[x]$. The constant term of f is up to sign equal to $Nrd(b)$, so $Nrd(b) \in V$. If $b \in B^*$ then $Nrd(b^{-1}) \in V$, and $1 = Nrd(b)Nrd(b^{-1})$, so $Nrd(b) \in V^*$. Conversely, if $Nrd(b) \in V^*$, write $f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} + x^n$ (so $\alpha_0 = \pm Nrd(b)$). Since $\alpha_0 \in V^*$ we can write $1 = b[-\alpha_0^{-1}(\alpha_1 + \cdots + \alpha_{n-1} b^{n-2} + b^{n-1})]$. This shows $b^{-1} \in B$, so $b \in B^*$.

Proposition 2.6. *Let S be a central simple F -algebra, B a Dubrovin valuation ring of S integral over $V = B \cap F$ and w a value function on S with $B = B_w$. Then*

- (1) Set $v = w|_F$; v is a valuation on F corresponding to V .
- (2) $st(B) = \{s \in S^* \mid w(s) = \frac{1}{n}v(Nrd(s))\}$, where $n = \sqrt{[S:F]}$.
- (3) $w(s) \leq \frac{1}{n}v(Nrd(s))$ in general.
- (4) w is uniquely determined on S by $w|_F$ and B .

Proof. (1) As $F^* \subseteq st(B) = st(w)$ by Theorem 2.3, $w|_F = v$ is a valuation on F by Lemma 2.2. Clearly the valuation ring of v is $B \cap F$.

(2) For the inclusion $st(B) \subseteq \{s \in S^* \mid w(s) = \frac{1}{n}v(Nrd(s))\}$, if $s \in st(B)$, then since $|\Gamma_B : \Gamma_V|$ divides $[S:F] = n^2$, $s^{n^2} = \alpha u$ with $\alpha \in F$ and $u \in B^*$. Then as $s \in st(w)$, $n^2 w(s) = w(s^{n^2}) = w(\alpha) = v(\alpha)$, so $w(s) = \frac{1}{n^2}v(\alpha)$. Also, $n^2 v(Nrd(s)) = v(Nrd(s^{n^2})) = v(Nrd(\alpha)) + v(Nrd(u)) = nv(\alpha)$, thus $\frac{1}{n}v(Nrd(s)) = \frac{1}{n^2}v(\alpha) = w(s)$. For the reverse inclusion, take $s \in S^*$ with $w(s) = \frac{1}{n}v(Nrd(s))$. There is an $s_0 \in st(w)$ with $w(s) = w(s_0)$. Since $w(s_0) = \frac{1}{n}v(Nrd(s_0))$, we have $v(Nrd(ss_0^{-1})) = 0$, so $Nrd(ss_0^{-1}) \in V^*$. Since

$ss_0^{-1} \in B$ we see $ss_0^{-1} \in B^*$ by the above remark, so $s \in s_0 B^* \subseteq st(w)$. Thus $st(B) = \{s \in S^* \mid w(s) = \frac{1}{n}v(Nrd(s))\}$.

(3) Write $s = s_0 u$ with $s_0 \in st(B)$ and $u \in B - J(B)$. Then

$$\begin{aligned} \frac{1}{n}v(Nrd(s)) &= \frac{1}{n}v(Nrd(s_0 u)) = \frac{1}{n}v(Nrd(s_0)) + \frac{1}{n}v(Nrd(u)) \\ &\geq \frac{1}{n}v(Nrd(s_0)) = w(s_0) = w(s) \end{aligned}$$

since $Nrd(u) \in V$, hence $v(Nrd(u)) \geq 0$.

(4) The equality $st(B) = \{s \in S^* \mid w(s) = \frac{1}{n}v(Nrd(s))\}$ shows that w is uniquely determined on $st(B)$ by $w|_F = v$. If $s = s_0 u$ with $s_0 \in st(B)$ and $u \in B - J(B)$, then $w(s) = w(s_0)$ by (4), so w is uniquely determined on all of S . \square

3. APPLICATIONS

In this section we shall prove some results about integral Dubrovin valuation rings (that is, Dubrovin valuation rings integral over their center) by making heavy use of value functions. The similarity of value functions and valuations (in particular property (1) of Lemma 2.2) allow results about invariant valuation rings to be generalized to integral Dubrovin valuation rings. The style of argument used in [MW] is similar to the arguments below. By combining the Henselization theorem [W₂, Theorem B] with results about integral Dubrovin valuation rings we are able to obtain results about arbitrary Dubrovin valuation rings (see for instance Corollary 3.6).

Lemma 3.1. *Suppose B is an integral Dubrovin valuation ring of a central simple algebra S , F a subfield of $Z(S)$, $V = B \cap F$, and w is a value function on S corresponding to B with $w|_F = v$. Take $a_1, \dots, a_e \in st(B)$ such that the $w(a_i)$ are distinct mod Γ_V , and $u_1, \dots, u_f \in B - J(B)$ such that $\overline{u_1}, \dots, \overline{u_f}$ are linearly independent over \overline{V} . Then the elements $\{a_i u_j \mid 1 \leq i \leq e, 1 \leq j \leq f\}$ are linearly independent over F . Furthermore, if $x = \sum_{i,j} \alpha_{ij} a_i u_j$ with $\alpha_{ij} \in F$, then $w(x) = \min_{i,j} \{v(\alpha_{ij}) + w(a_i)\}$.*

Proof. Suppose $\alpha_{ij} \in F$. For a fixed i there is a j_0 with $v(\alpha_{ij_0}) \leq v(\alpha_{ij})$ for all j . Then $\alpha_{ij_0}^{-1} \alpha_{ij} \in V$ and $\sum_j \alpha_{ij_0}^{-1} \alpha_{ij} u_j = \sum_j \overline{\alpha_{ij_0}^{-1} \alpha_{ij}} \overline{u_j} \neq \overline{0}$ by the \overline{V} -independence of the $\overline{u_j}$. Therefore $\sum_j \alpha_{ij_0}^{-1} \alpha_{ij} u_j \in B - J(B)$, so $w(\sum_j \alpha_{ij} u_j) = v(\alpha_{ij_0}) = \min_j \{v(\alpha_{ij})\} \in \Gamma_V$. Now $w(\sum_j \alpha_{ij} a_i u_j) = w(a_i) + w(\sum_j \alpha_{ij} u_j) = w(a_i) + \min_j \{v(\alpha_{ij})\}$, since the $a_i \in st(B)$. As the $w(a_i)$ are distinct mod Γ_V , the elements $\min_j \{v(\alpha_{ij}) + w(a_i)\}$ are distinct, so $w(\sum_{i,j} \alpha_{ij} a_i u_j) = \min_{i,j} \{v(\alpha_{ij}) + w(a_i)\}$, by Lemma 2.2. This implies that if $\sum_{i,j} \alpha_{ij} a_i u_j = 0$ then all $\alpha_{ij} = 0$, so the $a_i u_j$ are linearly independent over F . \square

Given B a Dubrovin valuation ring of S lying over V , let $\varepsilon = \varepsilon_{B/V}$ be the number of elements in the set $\{\gamma \in \Gamma_B \mid 0 \leq \gamma < \alpha \ \forall \alpha \in \Gamma_V, \alpha > 0\}$. It is

easy to see that if $0 = \gamma_1 < \dots < \gamma_\varepsilon < \Gamma_V^+ = \{\alpha \in \Gamma_V \mid \alpha > 0\}$ then the γ_i are distinct mod Γ_V , so $\varepsilon_{B/V} \leq e_{B/V}$. It is well known (cf. [B, pp. 424, 427 or E, 13.5, 18.6]) that if K is a finite field extension of F and B is a valuation ring of K (lying over V), then B is a finite V -module iff $\varepsilon_{B/V} f_{B/V} = [K : F]$. This can be generalized to Dubrovin valuation rings:

Proposition 3.2. *Suppose B is a Dubrovin valuation ring of a central simple algebra S with $F = Z(S)$ and $V = B \cap F$. Then B is a finite V -module iff $\varepsilon_{B/V} f_{B/V} = [S : F]$. When this occurs, B is integral and defectless over V .*

Proof. Suppose $\varepsilon_{B/V} f_{B/V} = [S : F]$. Since $\varepsilon_{B,V} \leq e_{B,V}$, $e_{B,V} f_{B,V} = [S : F]$. Then B is integral and defectless by [W₂, Theorems C and F]. Let w be a value function on S corresponding to B . Choose $a_1, \dots, a_\varepsilon \in st(B)$ such that $\{w(a_i) \mid 1 \leq i \leq \varepsilon\}$ represents all the elements γ with $0 \leq \gamma < \Gamma_V^+$, and $u_1, \dots, u_f \in B - J(B)$ such that $\overline{u}_1, \dots, \overline{u}_f$ form a \overline{V} -basis of \overline{B} . Since the $w(a_i)$ are distinct mod Γ_V , Lemma 3.1 shows that the elements $a_i u_j$ are linearly independent over F , hence form an F -basis of S by the defectlessness of B/V . Take $x = \sum_{i,j} \alpha_{ij} a_i u_j \in B$, $\alpha_{ij} \in F$. Then $w(x) = \min_{i,j} \{v(\alpha_{ij}) + w(a_i)\} \geq 0$, so for all i, j , $v(\alpha_{ij}) + w(a_i) \geq 0$. If $v(\alpha_{ij}) < 0$ for some i, j then $0 < -v(\alpha_{ij}) \leq w(a_i)$ which would contradict the choice of the a_i . Therefore all $\alpha_{ij} \in V$, so $B \subseteq \sum_{i,j} V a_i u_j$. Since all $a_i u_j \in B$, $B = \sum_{i,j} V a_i u_j$, so B is a finite V -module.

Conversely, suppose B is a finite V -module. Since B is a torsion free V -module, B is free (as V is a valuation ring), and since $B \otimes_V F \cong S$, $\text{rank}_V(B) = [S : F]$. Let $s_1, \dots, s_\varepsilon \in st(B)$ be such that $\{s_1 B^*, \dots, s_\varepsilon B^*\} = \{\gamma \in \Gamma_B \mid 0 \leq \gamma < \alpha \forall \alpha \in \Gamma_B^+\}$ and $B = s_1 B \supset s_2 B \supset \dots \supset s_\varepsilon B \supset J(V)B$. Set $M_i = s_i B$ and $M_{\varepsilon+1} = J(V)B$. By definition of the s_i , $M_i J(B) \subseteq M_{i+1}$, so the M_i/M_{i+1} are (right) \overline{B} -modules. M_i/M_{i+1} is free as a \overline{B} -module of rank 1. To see this, clearly $\overline{s}_i = s_i + M_{i+1}$ generates M_i/M_{i+1} . If $\overline{s}_i \overline{b} = \overline{0}$ for $b \in B$, $s_i b \in M_{i+1}$, so $s_i b \in s_{i+1} B$. If $b \in B - J(B)$, we then have $B s_i b B = s_i B b B = s_i B \not\subseteq s_{i+1} B$, so $b \in J(B)$, hence $\overline{b} = \overline{0}$. Therefore we get

$$\begin{aligned} [S : F] &= \text{rank}_V(B) = \text{rank}_{\overline{V}}(B/J(V)B) = \sum_{i=1}^{\varepsilon} \text{rank}_{\overline{V}}(M_i/M_{i+1}) \\ &= \sum_{i=1}^{\varepsilon} [\overline{B} : \overline{V}] \text{rank}_{\overline{B}}(M_i/M_{i+1}) = \sum_{i=1}^{\varepsilon} f_{B/V} = \varepsilon_{B/V} f_{B/V}. \end{aligned}$$

Therefore $\varepsilon_{B/V} f_{B/V} = [S : F]$. \square

In dealing with central simple algebras one frequently has to consider tensor products of algebras and extension of scalars. When studying valuation theory it is very useful to be able to determine properties of Dubrovin valuation rings in a tensor product given information about each of the pieces. For invariant valuation rings, Theorem 1 of [M₁] has proven to be quite useful, as seen in

[JW₂]. In that result a function was defined and shown to be a valuation. The proof was quite long due to the difficulty in proving the multiplicative property $v(ab) = v(a) + v(b)$. The inequality $v(ab) \geq v(a) + v(b)$ was quite easy to show however. The following is a generalization to Dubrovin valuation rings of that result, and the idea of the proof is much the same, but is much easier due to the less restrictive multiplicative property of value functions on central simple algebras. Using the terminology of [W₂], if $S \subseteq T$ are central simple algebras with $B \subseteq S$ and $A \subseteq T$ Dubrovin valuation rings such that $A \cap S = B$, $J(B) \subseteq J(A)$ and $st(B) \subseteq st(A)$, then we say A is a *compatible* extension of B . Note that for a compatible extension A/B there are natural inclusions $\Gamma_B \hookrightarrow \Gamma_A$ and $\overline{B} \hookrightarrow \overline{A}$.

Theorem 3.3. *Let S_1 and S_2 be central simple algebras, $F \subseteq Z(S_i)$, $B_i \subseteq S_i$ integral Dubrovin valuation rings with $B_1 \cap F = B_2 \cap F = V$. Suppose*

- (1) $\Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$,
- (2) $\overline{B_1} \otimes_{\overline{V}} \overline{B_2}$ is simple Artinian,
- (3) B_1/V is defectless.

If B is a Dubrovin valuation ring in $S = S_1 \otimes_F S_2$ then B is integral, $\Gamma_B = \Gamma_{B_1} + \Gamma_{B_2}$, $\overline{B} = \overline{B_1} \otimes_{\overline{V}} \overline{B_2}$, $\delta(B) = \delta(B_2)$ and the matrix size of $S_1 \otimes_F S_2$ divides the matrix size of $\overline{B_1} \otimes_{\overline{V}} \overline{B_2}$. Furthermore, B can be chosen to be a compatible extension of B_1 and B_2 .

Proof. Since $|\Gamma_{B_1} : \Gamma_V| < \infty$ (by (3)), there is a unique order preserving homomorphism $\Gamma_{B_1} \hookrightarrow \Gamma_V \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \Gamma_{B_2} \otimes_{\mathbb{Z}} \mathbb{Q} = \Delta_{B_2}$, Δ_{B_2} the divisible hull of Γ_{B_2} . The intersection $\Gamma_{B_1} \cap \Gamma_{B_2}$ is then computed in Δ_{B_2} . Let $a_1, \dots, a_e \in st(\overline{B})$ be representatives of Γ_{B_1}/Γ_V and $u_1, \dots, u_f \in B - J(B)$ map to a \overline{V} -basis of $\overline{B_1}$ with $a_1 = u_1 = 1$. Since B_1/V is defectless the $a_i u_j$ are an F -basis of S_1 by Lemma 3.1. Thus every element in S can be written uniquely in the form $\sum_{i,j} a_i u_j \otimes s_{ij}$. Let w_i be a value function on S_i corresponding to B_i with $w_1|_F = w_2|_F = v$. Define w on S by

$$w \left(\sum_{i,j} a_i u_j \otimes s_{ij} \right) = \min_{i,j} \{w_1(a_i) + w_2(s_{ij})\}.$$

Let $B = \{s \in S \mid w(s) \geq 0\}$ and $J = \{s \in S \mid w(s) > 0\}$. We will show that w is a value function on S and B/J is simple Artinian, so by Theorem 2.4, B will be an integral Dubrovin valuation ring of S . We first show that

- (a) $w(s+t) \geq \min\{w(s), w(t)\}$,
- (b) $w(st) \geq w(s) + w(t)$,
- (c) $w(s_1 \otimes s_2) = w_1(s_1) + w_2(s_2)$.

Property (c) shows that $B_1 \otimes_V B_2 \subseteq B$ and $J(B_i) = B_i \cap J$, hence $\overline{B_1}, \overline{B_2} \hookrightarrow \overline{B/J}$. To demonstrate these properties, let

$$s = \sum_{i,j} a_i u_j \otimes s_{ij} \quad \text{and} \quad t = \sum_{i,j} a_i u_j \otimes t_{ij}.$$

(a) $w(s+t) = \min_{i,j} \{w_1(a_i) + w_2(s_{ij} + t_{ij})\}$. Suppose the minimum occurs at $i, j = k, l$, and that $w_2(s_{kl}) \leq w_2(t_{kl})$. Then

$$w(s+t) = w_1(a_k) + w_2(s_{kl} + t_{kl}) \geq w_1(a_k) + w_2(s_{kl}) \geq \min_{i,j} \{w_1(a_i) + w_2(s_{ij})\} \\ = w(s) \geq \min\{w(s), w(t)\}.$$

(c) Say $s_1 = \sum_{i,j} \alpha_{ij} a_i u_j$ with $\alpha_{ij} \in F$. Then $s_1 \otimes s_2 = \sum_{i,j} a_i u_j \otimes \alpha_{ij} s_2$, so

$$w(s_1 \otimes s_2) = \min_{i,j} \{w_1(a_i) + w_2(\alpha_{ij} s_2)\} \\ = \min_{i,j} \{w_1(a_i) + v(\alpha_{ij}) + w_2(s_2)\} = \min_{i,j} \{w_1(a_i) + v(\alpha_{ij})\} + w_2(s_2) \\ = w_1(s_1) + w_2(s_2).$$

(b) Let $\sigma_{ij} = a_i u_j \otimes s_{ij}$ and $\tau_{ij} = a_i u_j \otimes t_{ij}$. So $st = \sum_{i,j,k,l} \sigma_{ij} \tau_{kl}$. Then by (a) and (c),

$$w(st) \geq \min_{i,j,k,l} \{w(\sigma_{ij} \tau_{kl})\} \geq \min_{i,j,k,l} \{w(\sigma_{ij}) + w(\tau_{kl})\} = w(s) + w(t).$$

Since $\overline{B_i} \hookrightarrow B/J$ and their images commute there is a \overline{V} -algebra homomorphism $\overline{B_1} \otimes_{\overline{V}} \overline{B_2} \rightarrow B/J$ given by $\overline{b_1} \otimes \overline{b_2} \mapsto b_1 \otimes b_2 + J$. This map is injective by the simplicity of $\overline{B_1} \otimes_{\overline{V}} \overline{B_2}$. For surjectivity, let $b = \sum_{i,j} a_i u_j \otimes s_{ij} \in B - J$. So $w(b) = 0$, hence some $w_1(a_i) + w_2(s_{ij}) = 0$. Since $\Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$ this can only happen for $i = 1$ as $a_1 = 1$, and so all $w_2(s_{1j}) \geq 0$ and $w_2(s_{1j}) = 0$ for some j . As the elements $\sum_j a_i u_j \otimes s_{ij}$ have distinct w -values, $b + J = (\sum_j u_j \otimes s_{1j}) + J$. Thus $\sum_j \overline{u_j} \otimes \overline{s_{1j}} \rightarrow b + J$. Hence $B/J \cong \overline{B_1} \otimes_{\overline{V}} \overline{B_2}$, so is simple Artinian.

By (c), $st(w_1) \otimes st(w_2) \subseteq st(w)$ and it is clear from the definition of w that $im(w) = w(st(w_1) \otimes st(w_2))$. Hence the hypotheses of Theorem 2.4 are satisfied. So B is an integral Dubrovin valuation ring with w a value function on S corresponding to B . The definition of w shows $\Gamma_B = im(w) = \Gamma_{B_1} + \Gamma_{B_2}$ and the above argument gives $\overline{B} = \overline{B_1} \otimes_{\overline{V}} \overline{B_2}$. Since $J(B) \cap B_i = J(B_i)$ and $st(B_i) \subseteq st(B)$, B/B_i is a compatible extension. The matrix size of S divides that of $n_B = t_B$, which is the matrix size of $\overline{B} = \overline{B_1} \otimes_{\overline{V}} \overline{B_2}$ since B is integral. Finally to show $\delta(B) = \delta(B_2)$, by [W₂, Theorem C],

$$[S : Z(S)] = |\Gamma_B : \Gamma_{Z(B)}| [\overline{B} : \overline{Z(B)}] \delta(B), \\ [S_1 : Z(S_1)] = |\Gamma_{B_1} : \Gamma_{Z(B_1)}| [\overline{B_1} : \overline{Z(B_1)}], \\ [S_2 : Z(S_2)] = |\Gamma_{B_2} : \Gamma_{Z(B_2)}| [\overline{B_2} : \overline{Z(B_2)}] \delta(B_2).$$

Since $S = S_1 \otimes_F S_2$ and $Z(S) = Z(S_1) \otimes_F Z(S_2)$,

$$[S : Z(S)] = [S_1 : Z(S_1)] [S_2 : Z(S_2)].$$

Replacing S , S_1 and S_2 by $Z(S)$, $Z(S_1)$ and $Z(S_2)$ respectively and applying what has been proven, we see that $\Gamma_{Z(B)} = \Gamma_{Z(B_1)} + \Gamma_{Z(B_2)}$ and $\overline{Z(B)} = \overline{Z(B_1)} \otimes_{\overline{V}} \overline{Z(B_2)}$. This residue property along with $\overline{B} = \overline{B_1} \otimes_{\overline{V}} \overline{B_2}$ yields

$$[\overline{B} : \overline{Z(B)}] = [\overline{B_1} : \overline{Z(B_1)}] [\overline{B_2} : \overline{Z(B_2)}].$$

We note that $\Gamma_B = \Gamma_{B_1} + \Gamma_{B_2}$ and $\Gamma_{Z(B)} = \Gamma_{Z(B_1)} + \Gamma_{Z(B_2)}$ together with $\Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$ implies the isomorphism $\Gamma_B / \Gamma_{Z(B)} \cong \Gamma_{B_1} / \Gamma_{Z(B_1)} \oplus \Gamma_{B_2} / \Gamma_{Z(B_2)}$. This then gives

$$|\Gamma_B : \Gamma_{Z(B)}| = |\Gamma_{B_1} : \Gamma_{Z(B_1)}| |\Gamma_{B_2} : \Gamma_{Z(B_2)}|.$$

Putting all this together gives $\delta(B) = \delta(B_2)$. Since all Dubrovin valuation rings in S extending V are conjugate by $[W_2, \text{Theorem A}]$, the theorem is proven. \square

There are a number of occasions when the tensor product of Dubrovin valuation rings is again a Dubrovin valuation ring, as seen in $[W_2, \S 3]$. As seen in the above proof $B_1 \otimes_V B_2 \subseteq B$. The next result gives an easily checked condition on when $B_1 \otimes_V B_2$ is a Dubrovin valuation ring in $S_1 \otimes_F S_2$.

Theorem 3.4. *With the notation of Theorem 3.3, $B_1 \otimes_V B_2$ is a Dubrovin valuation ring iff any nonnegative element of $\Gamma_{B_1} + \Gamma_{B_2}$ can be written in the form $\gamma_1 + \gamma_2$ with $\gamma_i \in \Gamma_{B_i}$ and $\gamma_1, \gamma_2 \geq 0$.*

Proof. Let B be as in the proof of Theorem 3.3. Since $B_1 \otimes_V B_2 \subseteq B$ it follows from $[D_2, \S 2, \text{Theorem 1}]$ that $B_1 \otimes_V B_2$ is a Dubrovin valuation ring iff $B_1 \otimes_V B_2 = B$. For an ordered group Γ , set $\Gamma^\bullet = \{\gamma \in \Gamma \mid \gamma \geq 0\} = \Gamma^+ \cup \{0\}$. The condition above can then be stated as $\Gamma_B^\bullet = \Gamma_{B_1}^\bullet + \Gamma_{B_2}^\bullet$. Suppose $\Gamma_B^\bullet = \Gamma_{B_1}^\bullet + \Gamma_{B_2}^\bullet$. Take $b = \sum_{i,j} a_i u_j \otimes s_{ij} \in B$. Then $w(b) \geq 0$, so for all i, j , $w_1(a_i) + w_2(s_{ij}) \geq 0$. Therefore there are $\alpha_{ij} \in st(B_1) \cap B_1$, $\beta_{ij} \in st(B_2) \cap B_2$ with $w_1(a_i) + w_2(s_{ij}) = w_1(\alpha_{ij}) + w_2(\beta_{ij})$. So $w_1(a_i) - w_1(\alpha_{ij}) = w_2(\beta_{ij}) - w_2(s_{ij}) \in \Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$. So there are $g_{ij} \in B_1 - J(B_1)$, $h_{ij} \in B_2 - J(B_2)$, $x_{ij} \in F^*$ with $a_i = \alpha_{ij} g_{ij} x_{ij}$, $s_{ij} = \beta_{ij} h_{ij} x_{ij}^{-1}$. Hence $a_i u_j \otimes s_{ij} = \alpha_{ij} g_{ij} u_j \otimes \beta_{ij} h_{ij} \in B_1 \otimes_V B_2$. So $b \in B_1 \otimes_V B_2$, hence $B = B_1 \otimes_V B_2$.

Conversely, suppose $B = B_1 \otimes_V B_2$. Take $b \in B = B_1 \otimes_V B_2$. We will show that $w(b) \in \Gamma_{B_1} + \Gamma_{B_2}$. Write $b = \sum_i d_i \otimes e_i$ with $d_i \in B_1$, $e_i \in B_2$, and say $d_i = \sum_{s,t} \alpha_{sti} a_s u_t$. Since $d_i \in B_1$, $w_1(d_i) \geq 0$, so $w_1(\alpha_{sti} a_s) \geq 0$ for all s, t, i . Also, $b = \sum_{s,t} a_s u_t \otimes (\sum_i \alpha_{sti} e_i)$. Pick s_0, t_0 with $w(b) = w_1(a_{s_0}) + w_2(\sum_i \alpha_{s_0 t_0 i} e_i)$ and i_0 with $v(\alpha_{s_0 t_0 i_0}) = \min_i \{v(\alpha_{s_0 t_0 i})\}$. Therefore $\alpha_{s_0 t_0 i_0}^{-1} \alpha_{s_0 t_0 i} \in V$, so $e = \sum_i \alpha_{s_0 t_0 i_0}^{-1} \alpha_{s_0 t_0 i} e_i \in B_2$. Hence $w(b) = w_1(\alpha_{s_0 t_0 i_0} a_{s_0}) + w_2(e) \in \Gamma_{B_1}^\bullet + \Gamma_{B_2}^\bullet$. Therefore $w(B) \subseteq \Gamma_{B_1}^\bullet + \Gamma_{B_2}^\bullet$. Since $w(B) = \Gamma_B^\bullet \supseteq \Gamma_{B_1}^\bullet + \Gamma_{B_2}^\bullet$, we get $\Gamma_B^\bullet = \Gamma_{B_1}^\bullet + \Gamma_{B_2}^\bullet$. \square

As an easy corollary of Theorem 3.3 we get $[M_1, \text{Theorem 1}]$.

Corollary 3.5. *With the notation above, suppose along with the hypotheses of Theorem 3.3 that S_1 and S_2 are division rings, B_1 and B_2 are invariant valuation rings and that $\overline{B_1} \otimes_{\overline{V}} \overline{B_2}$ is a division ring. Then $S_1 \otimes_F S_2$ is a division ring and B is an invariant valuation ring.*

Proof. Since the matrix size of $S_1 \otimes_F S_2$ divides the matrix size of $\overline{B_1} \otimes_{\overline{V}} \overline{B_2}$, $S_1 \otimes_F S_2$ is a division ring. Since \overline{B} is a division ring, $n_B = t_B = 1$

by $[W_2, \text{Theorem F}]$ as B is integral. Thus $[M_1, \text{Theorem 2}]$ shows that B is an invariant valuation ring. \square

The added generality of the residue condition (2) in Theorem 3.3 yields new information about division algebras over Henselian fields (beyond that given by Corollary 3.5). If (F, V) is a Henselian field, D_1, D_2 F -central division algebras with $V_{D_i} \subset D_i$ the invariant valuation rings lying over V and satisfying the hypotheses of Theorem 3.3, then if D is the underlying division algebra of $D_1 \otimes_F D_2$, then $\Gamma_D = \Gamma_{V_{D_1}} + \Gamma_{V_{D_2}}$ and $\overline{V_D} \sim \overline{V_{D_1}} \otimes_{\overline{V}} \overline{V_{D_2}}$. Furthermore it is not hard to see that the matrix size of $D_1 \otimes_F D_2$ is equal to the matrix size of $\overline{V_{D_1}} \otimes_{\overline{V}} \overline{V_{D_2}}$.

We can combine Theorem 3.3 with $[W_2, \text{Theorem B}]$ to get information about arbitrary Dubrovin valuation rings. For two central simple F -algebras C and C' , we write $C \sim C'$ when $[C] = [C']$ in $\text{Br}(F)$, that is when the underlying division algebras of C and C' are F -isomorphic.

Corollary 3.6. *Suppose S_1 and S_2 are central simple F -algebras, $B_i \subseteq S_i$ Dubrovin valuation rings lying over V with $\Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$, $\overline{B_1} \otimes_{\overline{V}} \overline{B_2}$ simple Artinian and B_1/V defectless. If B is a Dubrovin valuation ring in $S = S_1 \otimes_F S_2$ lying over V then $\Gamma_B = \Gamma_{B_1} + \Gamma_{B_2}$ and $\overline{B} \sim \overline{B_1} \otimes_{\overline{V}} \overline{B_2}$.*

Proof. Let (F_h, V_h) be the Henselization of (F, V) . Say $S_i \otimes_F F_h = M_{n_i}(D_i)$ and $S \otimes_F F_h = M_n(D)$ for D_1, D_2, D division algebras. Since F_h is Henselian there are invariant valuation rings R_1, R_2 and R in D_1, D_2 and D respectively lying over V_h . By $[W_2, \text{Theorem B}]$ we have $\Gamma_{B_i} = \Gamma_{R_i}$, $\Gamma_B = \Gamma_R$, $\overline{B_i} \sim \overline{R_i}$ and $\overline{B} \sim \overline{R}$. Thus $\Gamma_{R_1} \cap \Gamma_{R_2} = \Gamma_{V_h}$, $\overline{R_1} \otimes_{\overline{V_h}} \overline{R_2}$ is simple Artinian and R_1/V is defectless, since $\delta(R_1) = \delta(B_1) = 1$ by $[W_2, \text{Theorem C}]$. So by Theorem 3.3, if B' is a Dubrovin valuation ring in $D_1 \otimes_F D_2$, $\Gamma_{B'} = \Gamma_{R_1} + \Gamma_{R_2}$ and $\overline{B'} = \overline{R_1} \otimes_{\overline{V_h}} \overline{R_2}$. Since $S \otimes_F F_h \sim D_1 \otimes_F D_2$, if B_h is a Dubrovin valuation ring in $S \otimes_F F_h$, then $\Gamma_{B_h} = \Gamma_{B'}$ and $\overline{B_h} \sim \overline{B'}$ by $[W_2, \text{Corollary 3.5}]$. So applying $[W_2, \text{Theorem B}]$ again gives the result. \square

4. EXAMPLES

The situation is more complicated for nonintegral Dubrovin valuation rings. First of all one does not have a value function to use. Secondly, the nonintegrality of a Dubrovin valuation ring B means that there is an overring A such that the valuation ring $\tilde{V} = V/(J(A) \cap V)$ in $\overline{Z(A)}$ extends in more than one way to $Z(\overline{A})$ $[W_2, \text{Theorem F}]$. This gives room for complications to occur, as illustrated in the following examples. We first prove a proposition that will help us to construct these examples. Recall that an invariant valuation ring R of an F -central division algebra D is called *inertial* if $[\overline{R} : \overline{V}] = [D : F]$ and $Z(\overline{R}) = \overline{V}$, where $V = R \cap F$. It then follows that $\Gamma_R = \Gamma_V$ and that R/V is defectless.

Proposition 4.1. *Let S be a central simple F -algebra and V a valuation ring of F . Suppose A is an Azumaya algebra over V with quotient ring S and $\bar{A} = A/J(V)A$ a division ring. Then S is a division ring and A is an inertial invariant valuation ring of S .*

Proof. First assume F is Henselian, and say $S = M_n(D)$, where D is a division ring. By [JW₂, Proposition 2.5] $A \cong M_n(V_D)$, where V_D is the invariant valuation ring of D lying over V . Therefore $\bar{A} \cong M_n(\bar{V}_D)$, hence $n = 1$, so $S = D$ is a division ring and A is an invariant valuation ring. A is inertial also by [JW₂, Proposition 2.5]. In general, let (F_h, V_h) be the Henselization of (F, V) , and set $S_h = S \otimes_F F_h$ and $A_h = A \otimes_V V_h$. Then A_h is Azumaya over V_h and $\bar{A}_h = \bar{A}$. So by the above argument A_h is an inertial valuation ring in the division algebra S_h . Therefore by [M₁, Theorem 2], S is a division algebra and $A = A_h \cap S$ is an inertial valuation ring. \square

This proposition will be used as follows: Let S be the quaternion algebra $\left(\frac{a,b}{F}\right)$ for $a, b \in F^*$. This is the F -algebra with generators i, j and relations $i^2 = a, j^2 = b$ and $ij = -ji$ (see [L, Chapter 3] for more on quaternion algebras). Suppose that $a, b \in V^*$ and $\left(\frac{\bar{a}, \bar{b}}{\bar{V}}\right)$ is a division ring. The subring $A = \left(\frac{a,b}{V}\right)$ of S is an Azumaya algebra over V (see [JW₂, §2]) with $\bar{A} = \left(\frac{\bar{a}, \bar{b}}{\bar{V}}\right)$ so the proposition shows A is an inertial invariant valuation ring and so S is a division ring.

The following examples make heavy use of property (4) of §1: if A is a Dubrovin valuation ring of a central simple algebra S and \mathcal{B} is a Dubrovin valuation ring of \bar{A} then $B = \pi_A^{-1}(\mathcal{B})$ is a Dubrovin valuation ring of S , where $\pi_A: A \rightarrow \bar{A}$ is the canonical projection. If $F = Z(S)$, $V = B \cap F$ and $W = A \cap F$, then $\tilde{V} = V/J(W)$ is a valuation ring of \bar{W} . Following [W₂] we set $l_{B,A}$ equal to the number of extensions of \tilde{V} to $Z(\bar{A})$.

Example 4.2. D, D_1, D_2 F -central division algebras with $D = D_1 \otimes_F D_2$, B, B_1, B_2 Dubrovin valuation rings in D, D_1, D_2 respectively lying over V with B_1 integral, $\Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$ and $\bar{B}_1 \otimes_{\bar{V}} \bar{B}_2$ simple Artinian, but $\bar{B} \neq \bar{B}_1 \otimes_{\bar{V}} \bar{B}_2$.

Proof. Let

$$F = \mathbf{Q}(x, y), \quad D_1 = \left(\frac{-3, x+y}{F} \right), \quad D_2 = \left(\frac{-1, x}{F} \right)$$

and $D = D_1 \otimes_F D_2$. If X is the composite of the x -adic valuation ring of $\mathbf{Q}(y)(x)$ with the y -adic valuation ring of $\mathbf{Q}(y)$ and v is the corresponding valuation then (since $[\bar{X}(\sqrt{-3}, \sqrt{-1}) : \bar{X}] = 4$ and $v(x+y) = (1, 0), v(x) = (0, 1)$); by [JW₂, Ex. 4.3] D is a division ring.

Let $U = \mathbf{Q}(x)[y]_{(y)}$, the y -adic valuation ring of F and W the composite of U with the x -adic valuation ring on $\mathbf{Q}(x) = \bar{U}$. So $\bar{W} = \mathbf{Q}$. Let V be the

composite of W and the 5-adic valuation ring on Q . Thus we have

$$\begin{aligned}\Gamma_U &= Z, & \overline{U} &= Q(x), \\ \Gamma_W &= Z \times Z, & \overline{W} &= Q, \\ \Gamma_V &= Z \times Z \times Z, & \overline{V} &= Z/5Z.\end{aligned}$$

In D_1 , again by [JW₂, Ex. 4.3], V extends to a nicely semiramified valuation ring B_1 , as -3 is not a square mod 5 and $x + y$ does not have even value in Γ_V . Thus B_1 is integral (and defectless). For B_2 , we first see that W extends to a nicely semiramified valuation ring $A_2 \subset D_2$ with $\overline{A_2} = Q(\sqrt{-1})$. $V/J(W)$ (the 5-adic valuation ring of Q) extends in two ways to $Q(\sqrt{-1})$. If \mathcal{B}_2 is one of these extensions then $B_2 = \pi_{A_2}^{-1}(\mathcal{B}_2)$ is an extension of V to D_2 . Thus we have $\overline{B_2} = \overline{\mathcal{B}_2} = Z/5Z = \overline{V}$. Since $l_{B_2, A_2} = 2$ it follows from [W₂, Theorem C] that $n_{B_2} = 2$ and $\Gamma_{B_2} = \Gamma_V$. Therefore $\Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$ and $\overline{B_1} \otimes_{\overline{V}} \overline{B_2} = \overline{B_1}$ is simple.

To construct B we start with U . By Proposition 4.1, U extends to inertial valuation rings $C_i \subset D_i$ with $\overline{C_1} = \left(\frac{-3, x}{Q(x)}\right)$ and $\overline{C_2} = \left(\frac{-1, x}{Q(x)}\right)$. By Theorem 3.4 (or [W₂, Proposition 3.3]) $C = C_1 \otimes_U C_2$ is Dubrovin over U in D with $\overline{C} = \overline{C_1} \otimes_{\overline{U}} \overline{C_2} = M_2\left(\left(\frac{3, x}{Q(x)}\right)\right)$. $V/J(U)$ (the composite of the x -adic valuation ring of $Q(x)$ with $Z_{(5)}$) extends to a nicely semiramified valuation ring \mathcal{B} in $\left(\frac{3, x}{Q(x)}\right)$. Let $B = \pi_C^{-1}(M_2(\mathcal{B}))$. Then $B \subset D$ is (integral) Dubrovin over V with $\overline{B} = M_2(\overline{\mathcal{B}}) = M_2((Z/5Z)(\sqrt{3})) = M_2(\overline{B_1})$. Therefore $\overline{B} \neq \overline{B_1} \otimes_{\overline{V}} \overline{B_2} = \overline{B_1}$. \square

This shows the need for both B_1 and B_2 to be integral in Theorem 3.3. Notice also in the above example that U (and W) does not extend to an invariant valuation ring of D even though it does so extend to both D_1 and D_2 .

The next example shows that the tensor product of two central simple algebras need not contain an integral Dubrovin valuation ring even though each piece does. This example shows the necessity for assuming the simplicity of $\overline{B_1} \otimes_{\overline{V}} \overline{B_2}$ in Theorem 3.3. The example in [M₁] shows the need of defectlessness in at least one of the factors.

Example 4.3. D, D_1, D_2 F -central division algebras with $D = D_1 \otimes_F D_2$, $B_i \subset D_i$ integral Dubrovin valuation rings over V with $\Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$ but no Dubrovin valuation ring of D lying over V is integral.

Proof. Let

$$F = Q((x))((y)), \quad D_1 = \left(\frac{x, 3}{F}\right), \quad D_2 = \left(\frac{y, -3}{F}\right)$$

and $D = D_1 \otimes_F D_2$. D will be shown to be a division ring in much the same manner as in the previous example. Let U be the y -adic valuation ring

of F , $W = \mathbf{Q}[[x]] + y\mathbf{Q}((x))[[y]]$, the usual rank 2 valuation ring of F with $\overline{W} = \mathbf{Q}$, and V the composite of W and the 5-adic valuation ring of \mathbf{Q} . Notice that U and W are Henselian but V is not. By Proposition 4.1 and [JW₂, Ex. 4.3], U extends to an inertial valuation ring $C_1 \subset D_1$ with $\overline{C_1} = \left(\frac{x,3}{\mathbf{Q}((x))}\right)$ and to a nicely semiramified extension $C_2 \subset D_2$ with $\Gamma_{C_2} = \frac{1}{2}\mathbf{Z}$ and $\overline{C_2} = \mathbf{Q}((x))(\sqrt{-3})$. Hence D_1 and D_2 are division rings. We show that $\overline{C_1} \otimes_{\overline{U}} \overline{C_2} = \left(\frac{x,3}{\mathbf{Q}((x))(\sqrt{-3})}\right)$ is a division ring, which by Corollary 3.5 shows that D is a division ring and U extends to an invariant valuation ring $C \subset D$ with $\Gamma_C = \frac{1}{2}\mathbf{Z}$ and $\overline{C} = \overline{C_1} \otimes_{\overline{U}} \overline{C_2}$. To see this, since $\sqrt{3} \notin \mathbf{Q}((x))(\sqrt{-3})$, with respect to the x -adic valuation $\left(\frac{x,3}{\mathbf{Q}((x))(\sqrt{-3})}\right)$ is a nicely semiramified division algebra, by [JW₂, Ex. 4.3]. Hence D is a division ring and U extends to an invariant valuation ring $C \subset D$ with $\Gamma_C = \frac{1}{2}\mathbf{Z}$ and $\overline{C} = \overline{C_1} \otimes_{\overline{U}} \overline{C_2}$.

Using [JW₂, Ex. 4.3] again we see that W extends to nicely semiramified valuation rings A , A_1 , $A_2 \subset D$, D_1 , D_2 respectively with

$$\begin{aligned}\Gamma_A &= \frac{1}{2}\mathbf{Z} \times \frac{1}{2}\mathbf{Z}, & \overline{A} &= \mathbf{Q}(\sqrt{3}, \sqrt{-3}), \\ \Gamma_{A_1} &= \frac{1}{2}\mathbf{Z} \times \mathbf{Z}, & \overline{A_1} &= \mathbf{Q}(\sqrt{3}), \\ \Gamma_{A_2} &= \mathbf{Z} \times \frac{1}{2}\mathbf{Z}, & \overline{A_2} &= \mathbf{Q}(\sqrt{-3}).\end{aligned}$$

The 5-adic valuation ring $\mathbf{Z}_{(5)}$ extends uniquely to $\mathbf{Q}(\sqrt{3})$ and $\mathbf{Q}(\sqrt{-3})$ but extends in two ways to $\mathbf{Q}(\sqrt{3}, \sqrt{-3}) = \mathbf{Q}(\sqrt{3}, \sqrt{-1})$ as 3 is not a square mod 5 but -1 is. Let T_i be the unique extension of $\mathbf{Z}_{(5)}$ to $\overline{A_i}$ and T one of the extensions to \overline{A} . Let B , B_1 , B_2 be the composite of A , A_1 , A_2 with T , T_1 , T_2 respectively. We then have $l_{B_1, A_1} = l_{B_2, A_2} = l_{A_1, C_1} = l_{A_2, C_2} = 1$ and $l_{B, A} = 2$, $l_{A, C} = 1$. Thus using [W₂, Theorem D], $s_{B_1} = s_{B_2} = 1$ but $s_B = 2$. Hence B_1 and B_2 are integral but B is not. Since ± 3 is not a square mod 5, by [JW₂, Ex. 4.3] B_1 and B_2 are nicely semiramified valuation rings with $\Gamma_{B_1} = \mathbf{Z} \times \frac{1}{2}\mathbf{Z} \times \mathbf{Z}$, $\Gamma_{B_2} = \mathbf{Z} \times \mathbf{Z} \times \frac{1}{2}\mathbf{Z}$, $\overline{B_1} = (\mathbf{Z}/5\mathbf{Z})(\sqrt{3})$, $\overline{B_2} = (\mathbf{Z}/5\mathbf{Z})(\sqrt{-3}) = (\mathbf{Z}/5\mathbf{Z})(\sqrt{3}) = \overline{B_1}$. Thus $\Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$ and $\overline{B_1} \otimes_{\overline{V}} \overline{B_2}$ is not simple. Hence the simplicity of the ring $\overline{B_1} \otimes_{\overline{V}} \overline{B_2}$ is needed in Theorem 3.3. \square

REFERENCES

- [A] S. A. Amitsur, *On central division algebras*, Israel J. Math. **12** (1972), 408–420.
- [Be] H. Benz, *Über eine Bewertungstheorie der Algebren und ihre Bedeutung für die Arithmetik*, Schriftenreihe der Inst. für Math., Heft 9, Berlin, 1961.
- [B] N. Bourbaki, *Commutative algebra*, Chapter 6, Valuations, Hermann, Paris, 1961.
- [BG] H. H. Brungs and J. Gräter, *Extensions of valuation rings in central simple algebras*, Trans. Amer. Math. Soc. (to appear).
- [Dr] P. Draxl, *Ostrowski's theorem for Henselian valued skew fields*, J. Reine Angew. Math. **354** (1984), 213–218.
- [DK] P. Draxl and M. Kneser, *SK₁ von Schiefkörpern*, Lecture Notes in Math., vol. 778, Springer-Verlag, Berlin, Heidelberg and New York, 1980.

- [D₁] N. I. Dubrovin, *Noncommutative valuation rings*, Trudy Moskov. Mat. Obshch. **45** (1982), 265–280; English transl., Trans. Moscow Math. Soc. **45** (1984), 273–287.
- [D₂] ———, *Noncommutative valuation rings in simple finite-dimensional algebras over a field*, Mat. Sb. **123** (165) (1984), 496–509; English transl., Math. USSR-Sb. **51** (1985), 493–505.
- [E] O. Endler, *Valuation theory*, Springer-Verlag, New York, 1972.
- [JW₁] B. Jacob and A. Wadsworth, *A new construction of noncrossed product algebras*, Trans. Amer. Math. Soc. **293** (1986), 693–721.
- [JW₂] ———, *Division algebras over Henselian fields*, J. Algebra (to appear).
- [J] N. Jacobson, *PI-algebras, an introduction*, Lecture Notes in Math., vol. 441, Springer-Verlag, Berlin, Heidelberg and New York, 1975.
- [K] A. Kupferoth, *Valuated division algebras and crossed products*, J. Algebra **108** (1987), 139–150.
- [L] T.-Y. Lam *The algebraic theory of quadratic forms*, Benjamin, Reading, Mass., 1973.
- [M₁] P. Morandi, *The Henselization of a valued division algebra*, J. Algebra **122** (1989), 232–243.
- [M₂] ———, *Valuation rings in division rings and central simple algebras*, Doctoral Dissertation, Univ. of California, San Diego, 1988.
- [MW] P. Morandi and A. Wadsworth, *Integral Dubrovin valuation rings*, Trans. Amer. Math. Soc. **315** (1989), 623–640.
- [P] V. P. Platonov, *The Tannaka-Artin problem and reduced K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), 227–261; English transl., Math. USSR-Izv. **10** (1976), 211–243.
- [PY] V. P. Platonov and V. I. Yanchevskii, *Dieudonné's conjecture on the structure of unitary groups over a division ring, and hermitian K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), 1266–1294; English transl., Math. USSR-Izv. **25** (1985), 573–599.
- [R] I. Reiner, *Maximal orders*, Academic Press, London, 1975.
- [S] O. F. G. Schilling, *The theory of valuations*, Math Surveys, no. 4, Amer. Math. Soc., Providence, R.I., 1950.
- [T] J.-P. Tignol, *Cyclic and elementary abelian subfields of Malcev-Neumann division algebras*, J. Pure Appl. Algebra **42** (1986), 199–220.
- [TA] J.-P. Tignol and S. A. Amitsur, *Kummer subfields of Malcev-Neumann division algebras*, Israel J. Math. **50** (1985), 114–144.
- [TW] J.-P. Tignol and A. Wadsworth, *Totally ramified valuations on finite dimensional division algebras*, Trans. Amer. Math. Soc. **302** (1987), 223–250.
- [V] J. Van Geel, *Places and valuations in noncommutative ring theory*, Lecture Notes in Pure and Appl. Math., no. 71, Dekker, New York, 1981.
- [W₁] A. Wadsworth, *Dubrovin valuation rings*, Proceedings of the NATO Workshop “Perspectives in Ring Theory” (F. van Oystaeyen and L. Le Bruyn, eds.), Antwerp, 1988, pp. 359–374.
- [W₂] ———, *Dubrovin valuation rings and Henselization*, Math. Ann. **283** (1989), 301–328.

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